430. Proposed by G. PAASWELL, New York City.

Revolve a circle about a chord (not a diameter). Select a system of rectilinear coördinates with this chord as one axis and the origin as the intersection of the chord and the circumference. Term this axis the z axis and pass a plane through the x (or y) axis. Find the area of this surface intercepted by this plane and the xz (or yz) plane.

MECHANICS.

346. Proposed by WILLIAM HOOVER, Columbus, Ohio.

Half the length of one of the equal parts of a uniform heavy string resting in equilibrium over a smooth horizontal indefinitely thin peg is cut off; determine the instantaneous change in the pressure on the peg.

347. Proposed by E. B. ESCOTT, Kansas City, Missouri.

A cord $ABCD$ is suspended from points A and D, which are 20 feet apart in horizontal distance. D is 4 feet lower than A. At B and C are suspended weights of 100 and 200 lbs. $AB = 8$ feet, $BC = 10$ feet, and $CD = 12$ feet. Find angles α, β, γ made by AB, BC, and CD, respectively, with the horizontal. Also find tensions T_1, T_2, and T_3 in AB, BC, and CD.

NUMBER THEORY.

265. Proposed by J. W. NICHOLSON, Louisiana State University.

If the roots of $x^4 - ax^2 + bx + c = 0$ are rational, prove that $4(a + yz) - 3(y + z)$ is a perfect square, y and z being any two roots of the equation.

245. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that $x^2 + y^2 = (a_1a_2\cdots a_n)^n$ has $4(n + 1)^m$ solutions in integers, in 2^{m+2} of which x and y are relatively prime, the a's being primes of the form $4k + 1$ and n a positive integer.

Note.—The proposer of this problem has changed it to read as above instead of the statement as previously published in the May, 1916, issue.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

469. Proposed by T. H. GRONWALL, New York City.

Show that the equation

$$f(x) \equiv 2ax^4 + (1 - b)x^3 + b(1 - b)x - 2ab = 0,$$

where $0 < b < 1$, $a > 0$ and $a^2 > b$, has only one positive root and that it lies between the roots of

$$g(x) \equiv x^2 - 2ax + b = 0.$$

SOLUTIONS BY D. R. CURTISS, Northwestern University.

I. By Descartes' Rule of Signs, $f(x)$ has only one positive root. Let x_1 be the smaller of the two roots of $g(x) = 0$, and x_2 be the larger. We shall have solved the problem if we establish the inequalities

$$f(x_1) < 0, \qquad f(x_2) > 0. \tag{1}$$

Let us now give b a constant value and let a vary from $\sqrt{b}$ to ∞. If X is either x_1 or x_2 we shall have $g(X) = X^2 - 2aX + b = 0$, and by differentiating with respect to a we obtain

$$\frac{dX}{da} = \frac{X}{X - a}. \tag{2}$$

We now have

$$\frac{df(X)}{da} = f'(X)\frac{dX}{da} + 2(X^4 - b) = \frac{f'(X)\cdot X + 2(X - a)(X^4 - b)}{X - a}$$

or, in expanded form,

$$\text{(3)} \qquad \frac{df(X)}{da} = \frac{2X^5 + 6aX^4 + 3(1-b)X^3 + b(1-b)X + 2ab - 2bX}{X-a}.$$

The roots of $g(x) = 0$ satisfy the relations $0 < x_1 < a < x_2 < 2a$, from which we can show that the numerator on the right side of (3) is positive for both values of X. In fact, all the terms are positive except the last; but if $X = x_1 < a$, we have $2ab - 2bX > 0$, so that the numerator is positive in this case, while if $X = x_2$ we have, since $b < a^2$, and $a < x_2 < 2a$,

$$\begin{aligned} 2X^5 + b(1-b)X - 2bX + 2ab &= (2X^5 - b^2X) + (2ab - bX), \\ &> (2a^4 - a^4)X + b(2a - 2a) > 0. \end{aligned}$$

Since $x_1 - a$ is negative and $x_2 - a$ is positive, we thus have

$$\frac{df(x_1)}{da} < 0, \qquad \frac{df(x_2)}{da} > 0.$$

It follows that $f(x_1)$ has its greatest value, and $f(x_2)$ its least value, when $a = \sqrt{b}$, in which case $x_1 = x_2 = a$, and $f(x_1) = f(x_2) = 0$. The relations (1) follow at once.

II. We will show that the positive root of $f(x) = 0$ (there can be only one, by Descartes' Rule) lies between $x = \sqrt{b}$ and $x = a$. The interval $[\sqrt{b}, a]$ is, however, wholly comprised between the two roots of $g(x) = 0$, since $g(0)$ and $g(\infty)$ are positive, while

$$g(\sqrt{b}) = 2b - 2a\sqrt{b} = 2\sqrt{b}(\sqrt{b} - a)$$

is negative, and $g(a) = b - a^2$ is also negative.

We now prove that $f(\sqrt{b}) < 0$ and $f(a) > 0$. In the first place,

$$f(\sqrt{b}) = 2ab^2 + 2b\sqrt{b}(1-b) - 2ab.$$

Since $\sqrt{b} < a$, the substitution of a for $\sqrt{b}$ in the second term above gives

$$f\sqrt{b} < 2ab^2 + 2ab(1-b) - 2ab = 0.$$

On the other hand, we have

$$f(a) = 2a^5 + a^3(1-b) + ab(1-b) - 2ab = 2a^5 + a^3 - ab(a^2 + 1 + b).$$

If we replace each b above by a^2 we obtain the inequality

$$f(a) > 2a^5 + a^3 - a^3(a^2 + 1 + a^2) = 0.$$

Also solved by E. J. MOULTON, J. E. ROWE, H. H. CONWELL, J. A. BULLARD, J. L. RILEY, HORACE OLSON, J. W. BALDWIN, and the PROPOSER.

470. Proposed by ERNEST W. BROWN, Yale University.

There are n numbers each lying between $-\frac{1}{2}$ and $+\frac{1}{2}$, such that any value of each between these limits is equally probable. What is the probability that their sum will lie between $s - \frac{1}{2}$ and $s + \frac{1}{2}$, where s is an integral multiple of $\frac{1}{2}$.

SOLUTION BY C. F. Gummer, Kingston, Ontario.

Consider the more general problem where the n numbers are chosen at random from the intervals $(a_1, b_1), \cdots (a_n, b_n)$, and the sum is to lie in the interval (a, b).

Let $c_1 = b_1 - a_1, \cdots, c_n = b_n - a_n$.

Let $f_n(x) \cdot dx$ be the probability that the sum of the first r numbers will lie between x and $x + dx$.

Then $f_1(x) = 1/c_1$ when $a_1 < x < b_1$, and zero in other cases, and

$$f_r(x) = \int_{x-b_r}^{x-a_r} \frac{f_{r-1}(\xi) \cdot d\xi}{c_r}.$$